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Reverse Hölder Inequalities and Approximation Spaces

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We develop a simple geometry free context where one can formulate and prove general forms of Gehring's Lemma. We show how our result follows from a general inverse type reiteration theorem for approximation spaces.

Key Words: maximal functions; distribution function; approximation spaces; interpolation.

1. INTRODUCTION

Reverse Hölder inequalities play an important role in the theory of weighted norm inequalities for classical operators and PDEs. Recall that given a fixed cube[2] Q in R^n, and $1<p<\infty$, we say that a nonnegative measurable function $w \in L^p(Q)$ satisfies a Reverse Hölder inequality ($w \in RH_p(Q) = RH_p$) if there exists $b \in (1, \infty)$ such that for all subcubes $Q' \subset Q$, we have

$$\frac{1}{|Q'|}\int_{Q'} w(x)^p \, dx \leqslant b \left(\frac{1}{|Q'|}\int_{Q'} w(x)\, dx \right)^p. \tag{1.1}$$

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[2] In what follows we only consider cubes with sides parallel to the coordinate axes.

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Reverse Hölder inequalities have a crucial self improving property discovered by Gehring ("Gehring's Lemma" cf. [6]-Lemma 3 page 270), namely if $w \in RH_p$ then there exists $\varepsilon = \varepsilon(w) > 0$ such that for $q \in (p, p+\varepsilon)$ it follows that $w \in L^q(Q)$, and moreover there exists a positive constant $c = c(p, b, n)$ such that

$$\left(\frac{1}{|Q|}\int_Q w(x)^q \, dx\right)^{1/q} \leqslant c \left(\frac{1}{|Q|}\int_Q w(x)^p \, dx\right)^{1/p}. \tag{1.2}$$

In other words $w \in RH_p \Rightarrow w \in RH_q$, for some $q(w) > p$.

Gehring's celebrated result plays an important role in the theory of quasiconformal mappings, weighted norm inequalities and its applications to PDEs and functional analysis (cf. [2, 9, 12], and the references quoted therein). In [17] reverse Hölder inequalities were studied using real interpolation by means of reinterpreting the Condition (1.1) in terms of Peetre's K-functionals as follows

$$\frac{K(t^{1/p}, w; L^p(Q), L^\infty(Q))}{t^{1/p}} \leqslant c \, \frac{K(t, w; L^1(Q), L^\infty(Q))}{t}, \qquad 0 < t < |Q|. \tag{1.3}$$

In [17] it was then shown that (1.3) has a self improving property that leads to (1.2). In this manner Gehring's Lemma can be understood in the general setting of interpolation theory as a sort of inverse reiteration theorem. The K-functional approach is a natural extension of the classical proof of Gehring's Lemma, based on Calderón–Zygmund decompositions and the Hardy–Littlewood maximal operator. Indeed, recall that

$$\frac{K(t, f; L^1(Q), L^\infty(Q))}{t} \asymp (Mf)^*(t), \qquad 0 < t < |Q|,$$

where M is the usual maximal operator of Hardy–Littlewood associated with Q,

$$Mf(x) = \sup_{x \in Q' \subset Q} \frac{1}{|Q'|}\int_{Q'} |f(y)| \, dy.$$

Moreover, it is of interest to note that the K-functional method can be exploited to give new higher integrability results even when the Hardy–Littlewood maximal operator is not well behaved (cf. [14, 16]). For example, if the underlying measure is not doubling the corresponding K-functionals are equivalent to rearrangements of maximal operators associated with packings (cf. [1, 16]), which, unlike the corresponding maximal operator of Hardy–Littlewood, are bounded on L^p.

Reverse Hölder inequalities have been studied in a number of different contexts (e.g. "parabolic type", "homogeneous spaces", etc.) and in each case the underlying geometric considerations must be adapted accordingly. Interest in alternative formulations of RH_p inequalities also comes from recent research on vector valued weights (cf. [11, 22]).

These consideration have led us to investigate "geometry free" contexts where one can define and study reverse Hölder inequalities which have Gehring's self improving property. Note that (1.3) can be considered as a somewhat complicated "geometry free", definition of $RH_p(Q)$. In this note we consider simpler, "geometry free" formulations of reverse Hölder inequalities which are associated with the theory of approximation spaces. In the classical setting a formulation of our conditions can be given as follows,

$$\frac{\int_\Omega [w(x)-t]_+^p \, d\mu(x)}{t^p} \leqslant C \frac{\int_\Omega [w(x)-t]_+ \, d\mu(x)}{t}, \qquad t \geqslant t_0 \tag{1.4}$$

for some $t_0 > 0$, where $[x]_+ = \max(x, 0)$, and C is a constant independent of t. Condition (1.4) can be thus seen as a variant of the Hardy–Littlewood–Polya order[3]. It is easy to prove that (1.4) has Gehring's self improving property, in fact we show the following (cf. Section 3 for a simple direct proof)

LEMMA 1. *Let (Ω, μ) be σ-finite non atomic measure space and let $w \in L^1(\Omega)$ be a nonnegative function such that for some $t_0 > 0$ (1.4) holds for all $t \geqslant t_0$. Then, there exists $b > 0$ such that*

$$\int_\Omega w(x)^p \, d\mu(x) \leqslant b \int_\Omega w(x) \, d\mu(x),$$

and moreover there exists $\varepsilon = \varepsilon(b, p)$, and $c = c(p, q)$ such that for $q \in (p, p+\varepsilon)$, $w \in L^q(\Omega)$ and,

$$\int_\Omega w(x)^q \, d\mu(x) \leqslant c \int_\Omega w(x)^p \, d\mu(x).$$

Furthermore, if $t_0 = (\int_\Omega w(x)^p \, d\mu(x))^{1/p}$ and (1.4) holds for all $t \geqslant t_0$, then

$$\int_\Omega w(x)^q \, d\mu(x) \leqslant c \left(\int_\Omega w(x)^p \, d\mu(x) \right)^{q/p}.$$

[3] Recall that $w \prec \rho$ iff and only for all $t > 0$ we have $\int_\Omega [w(x)-t]_+ \, d\mu(x) \leqslant \int_\Omega [\rho(x)-t]_+ \, d\mu(x)$ (cf. [7]).

We actually show that a proof of this result can be obtained by an analysis of ideas in Gehring's classical paper [6]. We were led to formulate our results comparing [6] (cf. also [13]) with the methods associated with the theory of approximation spaces and the "error of approximation" functional of Peetre and Sparr (cf. [3, 20]) and the K-functional approach in [17]. In Section 4 we prove generalized forms of Lemma 1 in the context of approximation spaces, emphasizing its connection with reiteration formulae of Holmstedt–Nilsson type (cf. [19])[4].

THEOREM 1. *Let $\bar{X}=(X_0, X_1)$ be a pair of c_j-quasi-normed Abelian groups and suppose that f satisfies a Gehring condition (i.e. $f \in G_{a,r}$[5]). Then there exists $\alpha' > \alpha$ such that for all $q > 0$,*

$$E(t, f; X_0, E_{\alpha', q}(\bar{X})) \leqslant \tilde{c} t^{\alpha'} E(t, f; t, X_0, X_1).$$

In other words, $f \in G_{a,r} \Rightarrow f \in G_{a+\varepsilon, q}$, $q > 0$.

Some of our results are new even in the classical case, in particular, although in the classical context reverse Hölder inequalities are usually not considered for $p < 1$, our formalism leads naturally to a suitable interpretation in terms of "reverse Chebyshev inequalities" (cf. Section 5 below.)

Gehring elements can be characterized directly in terms of indices (cf. [16]) in particular the following abstract analogue of the A_∞ condition will be shown below (cf. Section 4)

THEOREM 2. *An element f satisfies a $G_{\alpha,r}$ Gehring condition $\Leftrightarrow$ for all $\varepsilon > 0$ there exists $\gamma = \gamma(\varepsilon) > 1$ such that*

$$\frac{E(t, \gamma f; X_0, X_1)}{E(t, f; X_0, X_1)} < \frac{\varepsilon}{\gamma^\alpha}, \qquad t \geqslant t_0. \tag{1.5}$$

The connection of these results with BMO will be discussed elsewhere [15].

For the benefit of the reader in Section 2 we review Gehring's approach to Gehring's Lemma and show a number of equivalent formulations in terms of distribution function inequalities. This analysis leads to an elementary proof of Lemma 1 in Section 3.

In conclusion in presenting Gehring's theory in this general context we also hope that these ideas could be useful to people working in Approximation Theory.

[4] Since the Holmstedt–Nilsson formulae is of independent interest we give a simple direct proof (cf. Theorem 4 below)

[5] (cf. Definition 2 below)

2. GEHRING'S LEMMA AND DISTRIBUTION FUNCTION INEQUALITIES

Gehring's original ideas play a fundamental role in our development. Therefore we start by reviewing the relevant part of [6].

Let (Ω, μ) be a σ-finite non atomic measure space. The distribution function of a measurable function f is given by

$$\lambda_f(t) = \mu\{x \in \Omega;\ |f(x)| > t\}, \qquad (t \geqslant 0).$$

We first state a number of known elementary results which shall be used in what follows.

Lemma 2. *Let* $0 < q < \infty$, $t_0 > 0$, *and let* h *be a decreasing function, then*

1. *If* $\lim_{x\to\infty} h(x) = 0$ *and* $\int_{t_0}^{\infty} s^q\, d(-h(s)) < \infty$ *then* $\lim_{x\to\infty} x^q h(x) = 0$.
2. *If* $\int_{t_0}^{\infty} s^q h(s)\, ds < \infty$ *then* $\lim_{x\to\infty} x^{q+1} h(x) = 0$.

The next result is well known (cf. [23]) we include a proof for the sake of completeness.

Lemma 3. *Let* (Ω, μ) *be a* σ*-finite non atomic measure space and* $w \in L^0(\Omega, \mu)$ *a weight (i.e. a nonnegative function) then*

$$\int_{\{w>t\}} w(s)^r\, d\mu(s) = t^r \lambda_w(t) + r \int_t^{\infty} s^{r-1} \lambda_w(s)\, ds, \qquad t, r > 0. \tag{2.1}$$

Define $h_w(t) = \int_{\{w>t\}} w(s)\, d\mu(s)$, and suppose that for some $t_0 > 0$ we have $h_w(t_0) < \infty$, then

$$\int_{\{w>t\}} w(s)^r\, d\mu(s) = \int_t^{\infty} s^{r-1} d(-h_w(s)), \qquad t > 0, \quad r \geqslant 1. \tag{2.2}$$

In particular, if $r \geqslant 1$, we have

$$\int_{\{w>t\}} w(s)^r\, d\mu(s) = \int_t^{\infty} s^{r-1} d(-h_w(s)) = \int_t^{\infty} s^r d(-\lambda_w(s)). \tag{2.3}$$

Proof. (2.1) follows immediately from the well known

$$\int_{\{w>t\}} w(s)^r\, d\mu(s) = \int_{\Omega} \underbrace{w(s)^r \chi\{w>t\}(s)}_{W^r}\, d\mu(s)$$
$$= r \int_0^{\infty} s^{r-1} \mu_W(s)\, ds.$$

Observe that (2.2) is obvious for $r=1$. Suppose that $r>1$, then

$$\int_{\{w>t\}} w(s)^r\, d\mu(s) = \int_{\Omega} \underbrace{w(s)^{r-1}\, \chi\{w>t\}(s)}_{W^{r-1}}\, \underbrace{w(s)\, ds}_{d\bar{\mu}(s)}$$

$$= (r-1) \int_0^{\infty} s^{r-2} \lambda_W(s)\, ds.$$

A simple computation shows that

$$\lambda_W(s) = \begin{cases} h_w(t) & \text{if} \quad s<t \\ h_w(s) & \text{if} \quad s \geqslant t' \end{cases}$$

thus

$$\int_{\{w>t\}} w(s)^r\, d\mu(s) = (r-1) \int_0^t s^{r-2} h_w(t)\, ds + (r-1) \int_t^{\infty} s^{r-2} h_w(s)\, ds.$$

The result follows integrating by parts the second integral appearing on the right hand side. It remains to prove the second inequality in (2.3) which we obtain integrating by parts the second integral on the right hand side of (2.1). ▮

The next basic elementary real variable result is due to Gehring,

Lemma 4 ([6] Lemma 1 p. 266). *Suppose that $p \in (0, \infty)$, $a \in (1, \infty)$ and let $t_0 > 0$. Suppose that h: $[t_0, \infty) \to [0, \infty)$ is decreasing with* $\lim_{t\to\infty} h(t) = 0$, *and that $\int_t^{\infty} s^p d(-h(s)) \leqslant a t^p h(t)$ for $t \in [t_0, \infty)$. Then*

$$\int_{t_0}^{\infty} s^q d(-h(s)) \leqslant \frac{p}{ap-(a-1)\,q}\, t_0^{q-p} \int_{t_0}^{\infty} s^p d(-h(s)),$$

for $q \in [p, pa/(a-1))$.

We now briefly review the main steps in Gehring's proof of (1.2). Suppose that $w \in RH_p$, we want to show that there exists $q>p$ such that $w \in RH_q$. Gehring shows that w satisfies the following estimate (cf. [6], p. 268-(12))

$$\int_{\{w>t\}} w(s)^p\, ds \leqslant c t^{p-1} \int_{\{w>t\}} w(s)\, ds, \qquad t \geqslant t_0, \tag{2.4}$$

where $t_0 = (\frac{1}{|Q|} \int_Q w(x)^p\, dx)^{1/p}$.

Using Lemma 3 it follows that (2.4) is equivalent to

$$\int_t^{\infty} s^{p-1} d(-h_w(s)) \leqslant c t^{p-1} h_w(t), \qquad t \geqslant t_0. \tag{2.5}$$

Lemma 4 now implies that we can choose $q > p$ such that

$$\int_{t_0}^{\infty} s^{q-1} d(-h_w(s)) \leqslant c_{p,q} t_0^{q-p} \int_0^{\infty} s^{p-1} d(-h_w(s)), \tag{2.6}$$

where $c_{p,q} = \frac{p-1}{c(p-1)-(c-1)(q-1)}$. Write

$$\int_Q w(x)^q \, dx = \int_{\{w > t_0\}} w(s)^q \, ds + \int_{\{w \leqslant t_0\}} w(s)^q \, ds. \tag{2.7}$$

By (2.6),

$$\begin{aligned} \int_{\{w > t_0\}} w(s)^q \, ds &= \int_{t_0}^{\infty} s^{q-1} d(-h_w(s)) \\ &\leqslant c_{p,q} t_0^{q-p} \int_{t_0}^{\infty} s^{p-1} d(-h_w(s)) \, ds \\ &= c_{p,q} t_0^{q-p} \int_{\{w > t_0\}} w(s)^p \, ds, \end{aligned}$$

while we obviously have

$$\int_{\{w \leqslant t_0\}} w(s)^q \, ds = \int_{\{w \leqslant t_0\}} w(s)^p \, w(s)^{q-p} \, ds \leqslant t_0^{q-p} \int_{\{w \leqslant t_0\}} w(s)^p \, ds.$$

Inserting these estimates in (2.7) we get

$$\int_Q w(x)^q \, dx \leqslant c_{p,q} t_0^{q-p} \int_Q w(x)^p \, dx.$$

Since $t_0 = (\frac{1}{|Q|} \int_Q w(x)^p \, dx)^{1/p}$ it follows that

$$\frac{1}{|Q|} \int_Q w(x)^q \, dx \leqslant c_{p,q} \left(\frac{1}{|Q|} \int_Q w(x)^p \, dx \right)^{q/p},$$

as we wished to show.

3. A GEOMETRY FREE VERSION OF GEHRING'S LEMMA

Motivated by the discussion in the previous section we introduce the following

DEFINITION 1. Let (Ω, μ) be a σ-finite non atomic measure space and let w be a nonnegative measurable function. Given $1 < p < \infty$ we shall say that $w \in G_p$ if there exist $c > 1$, $t_0 > 0$, such that for all $t \geqslant t_0$ it holds

$$\frac{\int_\Omega [w(x)-t]_+^p \, d\mu(x)}{t^p} \leqslant c \, \frac{\int_\Omega [w(x)-t]_+ \, d\mu(x)}{t}.$$

Remark 1. The connection with the Hardy–Littlewood–Polya order goes somewhat deeper. As is well know the HLP theory has been extended to measures (cf. [21] and the references therein), following the analogy further in this direction we say that a positive measure μ supported on $(0, \infty)$ satisfies a G_p condition if there exist $c > 1$ such that for all $t > 0$ we have

$$\frac{\int_\Omega [x-t]_+^p \, d\mu(x)}{t^p} \leqslant c \, \frac{\int_\Omega [x-t]_+ \, d\mu(x)}{t}.$$

If we associate to a given weight w the Lebesgue-Stieltjes measure $d\lambda_w$ generated by its distribution function we recover Definition 1. For more general measure spaces we should replace the test class of extremal "angle functions" $[x-t]_+^p$ by other suitable classes of convex functions. We shall give more details on this elsewhere.

The next result establishes the equivalence between G_p and (2.4) and several other related conditions. As we shall see below (cf. Theorem 6) the result can be suitably reinterpreted as a reiteration theorem for Gehring conditions!

THEOREM 3. *The following statements are equivalent*

1. $w \in G_p$.
2. There exists $c_0 > 1$ such that for all $t \geqslant t_0$,

$$\int_t^\infty s^{p-1} \lambda_w(s) \, ds \leqslant c_0 t^{p-1} \int_t^\infty \lambda_w(s) \, ds.$$

3. There exists $c_1>1$ such that for all $t \geqslant t_0$,

$$\int_{\{w>t\}} w(s)^p \, d\mu(s) \leqslant c_1 t^{p-1} \int_{\{w>t\}} w(s) \, d\mu(s).$$

4. There exists $c_2>1$ such that for all $t \geqslant t_0$,

$$\int_t^\infty s^{p-1} d(-h_w(s)) \leqslant c_2 t^{p-1} h_w(t).$$

5. There exists $c_3>1$ such that for all $t \geqslant t_0$,

$$\int_t^\infty s^{p-1} \lambda_w(s) \, ds \leqslant c_3 t^p \lambda_w(t).$$

Proof. $1 \Rightarrow 2$ Recall that

$$\int_\Omega [w(x)-t]_+^p \, d\mu = p \int_t^\infty (s-t)^{p-1} \lambda_w(s) \, ds.$$

If $c_p t = (\frac{1}{1-2^{-1/p-1}}) \, t \leqslant s$ then $(s-t)^{p-1} \geqslant \frac{1}{2} s^{p-1}$, therefore,

$$\begin{aligned} p \int_{c_p t}^\infty s^{p-1} \lambda_w(s) \, ds &\leqslant 2 \int_\Omega [w(x)-t]_+^p \, d\mu(x) \\ &\leqslant 2ct^{p-1} \int_\Omega [w(x)-t]_+ \, d\mu(x) \quad (\text{since } w \in G_p, \, t \geqslant t_0) \\ &= 2ct^{p-1} \int_t^\infty \lambda_w(s) \, ds. \end{aligned}$$

Adding $p \int_t^{c_p t} s^{p-1} \lambda_w(s) \, ds$ to both sides of the previous inequality we obtain

$$\begin{aligned} p \int_t^\infty s^{p-1} \lambda_w(s) \, du &\leqslant 2ct^{p-1} \int_t^\infty \lambda_w(s) \, du + p \int_t^{c_p t} s^{p-1} \lambda_w(s) \, ds \\ &\leqslant t^{p-1} (2c + pc_p^{p-1}) \int_t^\infty \lambda_w(s) \, ds \qquad (t \geqslant t_0), \end{aligned}$$

and 2 follows with $c_0 = (2c + pc_p^{p-1})/p$.

$2 \Rightarrow 3$ By Lemma 3 (2.1),

$$\begin{aligned}\int_{\{w>t\}} w(s)^p \, d\mu(s) &= t^p \lambda_w(t) + p \int_t^\infty s^{p-1} \lambda_w(s)\, ds \\ &\leqslant t^p \lambda_w(t) + p c_0 t^{p-1} \int_t^\infty \lambda_w(s)\, ds \qquad \text{(applying 2)} \\ &\leqslant p c_0 t^{p-1} \left(t\lambda_w(t) + \int_t^\infty \lambda_w(s)\, ds \right) \\ &= p c_0 t^{p-1} \int_{\{w>t\}} w(s)\, d\mu(s), \qquad \text{(by Lemma 3(2.1))}\end{aligned}$$

and we have obtained 3 with $c_1 = pc_0$.

$3 \Rightarrow 4$ By Lemma 3 (2.2),

$$\begin{aligned}\int_t^\infty s^{p-1} d(-h_w(s)) &= \int_{\{w>t\}} w(s)^p \, d\mu(s) \\ &\leqslant c_1 t^{p-1} \int_{\{w>t\}} w(s)\, d\mu(s) = c_1 t^{p-1} h_w(t).\end{aligned}$$

$4 \Rightarrow 5$ Applying Lemma 3 (2.3 and 2.1) twice, we see that Condition 4 can be rewritten as

$$\begin{aligned}t^p \lambda_w(t) + p \int_t^\infty s^{p-1} \lambda_w(s)\, ds &\leqslant c_2 t^{p-1} \int_{\{w>t\}} w(s)\, d\mu(s) \\ &= c_2 t^{p-1} \left(t\lambda_w(t) + \int_t^\infty \lambda_w(s)\, ds \right),\end{aligned}$$

thus

$$\int_t^\infty (ps^{p-1} - c_2 t^{p-1})\, \lambda_w(s)\, ds \leqslant (c_2 - 1)\, t^p \lambda_w(t).$$

If $s \geqslant c_2^{1/(p-1)} t = C_p t$, it follows that $ps^{p-1} - c_2 t^{p-1} \geqslant (p-1)\, s^{p-1}$, and therefore we have

$$\begin{aligned}\int_{C_p t}^\infty s^{p-1} \lambda_w(s)\, ds &\leqslant \frac{1}{p-1} \int_t^\infty (ps^{p-1} - c_2 t^{p-1})\, \lambda_w(s)\, ds \\ &\leqslant \frac{c_2 - 1}{p-1}\, t^p \lambda_w(t). \qquad (3.1)\end{aligned}$$

Now, we add $\int_t^{C_p t} s^{p-1}\lambda_w(s)\,ds$ to both sides of (3.1) and use the fact that λ_w is decreasing to obtain

$$\int_t^\infty s^{p-1}\lambda_w(s)\,ds \leqslant \left(\frac{c_2-1}{p-1}+\frac{c_2^{p/(p-1)}-1}{p}\right) t^p\lambda_w(t)=c_3 t^p\lambda_w(t).$$

$5 \Rightarrow 1$ Combine 5 with the fact that $\frac{t}{2}\lambda_w(t)\leqslant\int_{t/2}^\infty \lambda_w(s)\,ds$, to get

$$\int_t^\infty s^{p-1}\lambda_w(s)\,ds\leqslant 2c_3 t^{p-1}\int_{t/2}^\infty \lambda_w(s)\,ds.$$

Adding $\int_{t/2}^t s^{p-1}\lambda_w(s)$ to both sides of the inequality we get

$$\int_{t/2}^\infty s^{p-1}\lambda_w(s)\,ds\leqslant(2c_3+1)\,t^{p-1}\int_{t/2}^\infty \lambda_w(s)\,ds \qquad (t\geqslant t_0),$$

or equivalently,

$$\int_t^\infty s^{p-1}\lambda_w(s)\,ds\leqslant(2c_3+1)\,2^{p-1}t^{p-1}\int_t^\infty \lambda_w(s)\,ds \qquad (t\geqslant t_0),$$

but then

$$\begin{aligned}\int_\Omega [w(x)-t]_+^p\,d\mu(x)&=p\int_t^\infty (s-t)^{p-1}\,\lambda_w(s)\,ds\leqslant p\int_t^\infty s^{p-1}\lambda_w(s)\,ds\\ &\leqslant p(2c_3+1)\,2^{p-1}t^{p-1}\int_t^\infty \lambda_w(s)\,ds\\ &=p(2c_3+1)\,2^{p-1}t^{p-1}\int_\Omega [w(x)-t]_+\,d\mu(x). \quad \blacksquare\end{aligned}$$

We now give the short

Proof of Lemma 1. Suppose that w satisfies (1.4) for some $t_0>0$. Theorem 3 and Lemma 4 show that following the steps of Gehring's argument as outlined in the previous section we readily arrive to the desired result.

Remark 2 (See Lemma 5 below). Note that Gehring's Lemma 1 holds with the same proof if we modify Condition (1.4) to accommodate constants as follows: there exist $c>1$, γ_0, $\gamma_1>0$, $t_0>0$, such that for all $t\geqslant t_0$ it holds

$$\frac{\int_\Omega [w(x)-\gamma_1 t]_+^p\,d\mu(x)}{t^p}\leqslant c\,\frac{\int_\Omega [w(x)-\gamma_0 t]_+\,d\mu(x)}{t}$$

4. E-FUNCTIONAL APPROACH TO REVERSE HÖLDER INEQUALITIES

In this section we develop an approximation space approach to Gehring's Lemma, and in particular provide proofs of the results stated in the Introduction.

We shall start with a brief review of the necessary background on approximation spaces (for more information we refer to [3, 4, 19] and the references quoted therein).

In approximation theory it is important to consider spaces somewhat more general than Banach spaces. We indeed consider Abelian groups X equipped with c-quasi-norms $\|\cdot\|_X$, where $c \geqslant 1$, this means that $\|\cdot\|_X$ is a real valued function defined on X such that (note the lack of homogeneity)

1. $\|x\|_X \geqslant 0$, and $\|x\|_X = 0 \Leftrightarrow x = 0$
2. $\|x\|_X = \|-x\|_X$
3. $\|x+y\|_X \leqslant c(\|x\|_X + \|y\|_X)$.

Obviously every Banach space is a 1-quasi-norm Abelian group. Given a pair $\bar{X} = (X_0, X_1)$ of c_j-quasi-normed Abelian groups, and an element $f \in X_0 + X_1$, we let

$$E(t, f; X_0, X_1) = \inf_{\|f_0\|_{X_0} \leqslant t} \|f - f_0\|_{X_1}, \qquad 0 < t < \infty.$$

Throughout what follows $E(t, f) \equiv E(t, f; X_0, X_1)$.

It follows readily that $E(t, f)$ is a decreasing function of t and that for $0 < \varepsilon < 1$ we have (cf. [3])

$$E(t+g, f) \leqslant c_1(E(t, \varepsilon f/c_0) + E(\varepsilon, (1-g)\, t/c_0)). \tag{4.1}$$

It is also easy to see that if $E(t, f) = 0$ for all $t > 0$ then $f = 0$, and moreover if $f \in X_0$ with $\|f\|_{X_0} \leqslant t$ then $E(s, f) = 0$ for all $s \geqslant t$.

The approximation space $E_{\alpha, r}(X_0, X_1)$, $0 < \alpha$, $r < \infty$, consists of all $f \in X_0 + X_1$ such that

$$\|f\|_{E_{\alpha, r}(X_0, X_1)} = \left(\int_0^\infty (s^\alpha E(s, f))^r \frac{ds}{s} \right)^{1/r} < \infty.$$

Example 1 (cf. [20]). Let (Ω, μ) be a σ-finite non atomic measure space. The (Peetre–Sparr) space L^0 consists of all functions with finite support with the 1-quasi-norm given by

$$\|f\|_{L^0} = \mu(\{f \neq 0\}).$$

It is readily seen that (cf. [3, 20])

$$E(t, f; L^{\infty}, L^{0}) = \lambda_f(t), \tag{4.2}$$

and

$$E_{p,1}(L^{\infty}, L^{0}) \approx (L^{p})^{p}.$$

The next reiteration formula will play a crucial role in what follows.

THEOREM 4 (Holmstedt–Nilsson type formula (cf. [19])). *Let* $\bar{X} = (X_0, X_1)$ *be a pair of* c_j*-quasi-normed Abelian groups then*

$$\begin{aligned}
&\frac{1}{cc_1(2c_0)^{\alpha}} \left(\int_{2c_0 t}^{\infty} (s^{\alpha} E(s, f))^{r} \frac{ds}{s} \right)^{1/r} \\
&\quad \leqslant E(t, f; X_0; E_{\alpha, r}(\bar{X})) \\
&\quad \leqslant c(2c_2)^{\alpha} \left\{ t^{\alpha} E(t, f) + \left(\int_{t}^{\infty} (s^{\alpha} E(s, f))^{r} \frac{ds}{s} \right)^{1/r} \right\},
\end{aligned} \tag{4.3}$$

where $c = \min(1, 2^{(1-r)/r})$.

Proof. Let $g \in X_0$ with $\|g\|_{X_0} \leqslant t$, applying (4.1), with $\varepsilon = 1/2$, we get

$$\begin{aligned}
\left(\int_{2c_0 t}^{\infty} (s^{\alpha} E(s, f))^{r} \frac{ds}{s} \right)^{1/r} &\leqslant cc_1 \left(\int_{2c_0 t}^{\infty} \left(s^{\alpha} E\left(s - g, \frac{f}{2c_0} \right) \right)^{r} \frac{ds}{s} \right)^{1/r} \\
&\quad + cc_1 \left(\int_{2c_0 t}^{\infty} \left(s^{\alpha} E\left(s, \frac{g}{2c_0} \right) \right)^{r} \frac{ds}{s} \right)^{1/r}.
\end{aligned}$$

Note that since $\|g\|_{X_0} \leqslant t$ the second integral is 0, therefore

$$\begin{aligned}
\left(\int_{2c_0 t}^{\infty} (s^{\alpha} E(s, f))^{r} \frac{ds}{s} \right)^{1/r} &\leqslant cc_1(2c_0)^{\alpha} \left(\int_{0}^{\infty} (u^{\alpha} E(u - g, f))^{r} \frac{du}{u} \right)^{1/r} \\
&= cc_1(2c_0)^{\alpha} \|f - g\|_{E_{\alpha, r}(\bar{X})}.
\end{aligned}$$

Taking infimum over all $g \in X_0$ with $\|g\|_{X_0} \leqslant t$ the left-most inequality follows. The remaining inequality can be obtained as follows: for each $\delta > 0$, and $u > 0$ pick $f_u \in X_0$ with $\|f_u\|_{X_0} \leqslant u$ such that $\|f - f_u\|_{X_1} \leqslant (1 + \delta)\, E(u, f)$. Then,

$$\begin{aligned}\|f-f_t\|_{E_{\alpha,r}(X_0,X_1)} &= \left(\int_0^\infty (s^\alpha E(s-f_t,f))^r \frac{ds}{s}\right)^{1/r}\\ &\leqslant c\left[\left(\int_0^{2tc_1} (s^\alpha E(s-f_t,f))^r \frac{ds}{s}\right)^{1/r}\right.\\ &\quad\left.+\left(\int_{2tc_1}^\infty (s^\alpha E(s-f_t,f))^r \frac{ds}{s}\right)^{1/r}\right].\end{aligned} \tag{4.4}$$

Note that if $s \leqslant 2tc_1$ then

$$E(s-f_t,f) \leqslant \|f-f_t\|_{X_1} \leqslant (1+\delta)\, E(t,f),$$

while if $s > 2tc_1$ we have

$$\|f_{s/2c_1}-f_t\|_{X_1} \leqslant c_1(\|f_{s/2c_1}\|_{X_1}+\|f_t\|_{X_1}) \leqslant c_1(s/2c_1+t) \leqslant s,$$

consequently,

$$E(s-f_t,f) \leqslant \|f-f_t-(f_{s/2c_1}-f_t)\|_{X_1} \leqslant (1+\delta)\, E(f,s/2c_1).$$

Inserting these estimates in (4.4) we obtain

$$\begin{aligned}\|f-f_t\|_{E_{\alpha,r}(\bar{X})} &\leqslant c(1+\delta)\left[\left(\int_0^{2tc_1} (s^\alpha E(t,f))^r \frac{ds}{s}\right)^{1/r}\right.\\ &\quad\left.+\left(\int_{2tc_1}^\infty (s^\alpha E(2,s/fc_1))^r \frac{ds}{s}\right)^{1/r}\right]\\ &\leqslant c(1+\delta)(2c_1)^\alpha\left[t^\alpha E(t,f)+\left(\int_t^\infty (u^\alpha E(u,f))^r \frac{du}{u}\right)^{1/r}\right].\end{aligned}$$

Finally taking infimum over all $f_t \in X_0$ with $\|f_t\|_{X_0} \leqslant t$ we get

$$\begin{aligned}&E(t,f;X_0,E_{\alpha,r}(\bar{X}))\\ &\quad\leqslant c(1+\delta)(2c_1)^\alpha\left\{t^\alpha E(t,f)+\left(\int_t^\infty (u^\alpha E(u,f))^r \frac{du}{u}\right)^{1/r}\right\}.\end{aligned}$$

Letting $\delta \to 0$ the desired result follows. ∎

Remark 3. Since $(t^\alpha E(f,t))^r \leqslant C\int_{t/2}^t (s^\alpha E(s,f))^r \frac{ds}{s} \leqslant C\int_{t/2}^\infty (s^\alpha E(s,f))^r \frac{ds}{s}$ the upper estimate in the previous lemma can be rewritten as

$$E(t,f;X_0,E_{\alpha,r}(\bar{X})) \leqslant \tilde{C}\left(\int_{t/2}^\infty (s^\alpha E(s,f))^r \frac{ds}{s}\right)^{1/r}. \tag{4.5}$$

Consider the pair (L^∞, L^1) and let $f=\chi_{(0,1)}$. It is then easy to see that although we can replace "$2t$" by "ct", with $c>1$, on the left hand side of (4.3), the formula does not hold for "t".

Remark 4. Notice that (cf. [4, Proposition 3.1.16; 10])

$$E(t, f; L^\infty, L^p)^p = \int_\Omega [f(x)-t]_+^p \, d\mu(x).$$

Thus Condition (1.4) is equivalent to

$$E(t, f; L^\infty, L^p)^p \leqslant ct^{p-1} E(t, w; L^\infty, L^1).$$

Moreover since $E_{p-1,1}(L^\infty, L^1) = (L^p)^p$ (cf. [3, Corollary 7.2.3]) and

$$E(t, f; L^\infty, (L^p)^p) = \inf_{\|f_0\|_{L^\infty} \leqslant t} \|f-f_0\|_{L^p}^p = \Big(\inf_{\|f_0\|_{L^\infty} \leqslant t} \|f-f_0\|_{L^p}\Big)^p$$
$$= E(t, f; L^\infty, L^p)^p,$$

we can rewrite Condition (1.4) as

$$E(t, w; L^\infty, E_{p-1,1}(L^\infty, L^1)) \leqslant ct^{p-1} E(t, w; L^\infty, L^1).$$

The discussion in the previous remark motivates the following

DEFINITION 2. Let $\bar{X}=(X_0, X_1)$ be a pair of c_j-quasi-normed Abelian groups, and let α, $r>0$. We will say that $w \in X_0+X_1$ satisfies a Gehring (α, r)-condition (briefly $f \in G_{a,r} = G_{a,r}(\bar{X})$) if there exists $c>0$, $t_0>0$, such that

$$E(t, w; X_0, E_{\alpha,r}(\bar{X})) \leqslant ct^\alpha E(t, w), \qquad t \geqslant t_0 > 0. \tag{4.6}$$

The next Lemma will be useful in what follows.

LEMMA 5. *Let $\bar{X}=(X_0, X_1)$ be a pair of c_j-quasi-normed Abelian groups, let $w \in X_0+X_1$ and let γ_0, γ_1, $t_0>0$. Then,*

1. $w \in G_{a,r}(\bar{X}) \Leftrightarrow \big(\int_t^\infty (s^\alpha E(s,w))^r \frac{ds}{s}\big)^{1/r} \leqslant c_r t^\alpha E(t,w)$, $t \geqslant t_0 > 0$.
2. $E(t, \gamma_0 w; X_0, E_{\alpha,r}(\bar{X})) \leqslant ct^\alpha E(t, \gamma_1 w)$, $t \geqslant t_0 > 0 \Rightarrow w \in G_{a,r}(\bar{X})$.
3. $\big(\int_{\gamma_0 t}^\infty (s^\alpha E(s,w))^r \frac{ds}{s}\big)^{1/r} \leqslant ct^\alpha E(t, \gamma_1 w)$, $t \geqslant t_0 > 0 \Rightarrow w \in G_{a,r}(\bar{X})$.

Proof. 1. Suppose that $w \in G_{a,r}(\bar{X})$. Applying the Holmstedt–Nilsson formula with $D=c_1(2c_0)^\alpha \min(1, 2^{(1-r)/r})$ and $C=(2c_1)^\alpha \min(1, 2^{(1-r)/r})$, we see that for $t \geqslant t_0$ we have

$$\int_{2c_0 t}^\infty (s^\alpha E(s,w))^r \frac{ds}{s} \leqslant (cDt^\alpha E(t,w))^r.$$

Adding to both sides $\int_t^{2c_0 t} (s^\alpha E(s, w))^r \frac{ds}{s}$, using the fact that $E(s, f)$ is decreasing and collecting terms, we get

$$\left(\int_t^\infty (s^\alpha E(s, w))^r \frac{ds}{s}\right)^{1/r} \leqslant \left((cD)^r + \frac{(2c_0)^r - 1}{\alpha r}\right)^{1/r} t^\alpha E(t, w).$$

Conversely, adding to both sides of the hypothesized inequality $t^\alpha E(t, f)$ and multiplying by C

$$C\left(t^\alpha E(t, w) + \left(\int_t^\infty (s^\alpha E(s, w))^r \frac{ds}{s}\right)^{1/r}\right) \leqslant C(c' + 1)\, t^\alpha E(t, w),$$

and applying the Holmstedt–Nilsson formula we arrive to

$$E(t, w, X_0, E_{\alpha, r}(\bar{X})) \leqslant C(c' + 1)\, t^\alpha E(t, w).$$

2. By the Holmstedt–Nilsson formula we have

$$\int_{2c_0\gamma_0 t}^\infty (s^\alpha E(s, w))^r \frac{ds}{s} \leqslant (cDt^\alpha E(t, \gamma_1 w))^r.$$

If $0 < \gamma_0 \leqslant 1$ we can replace $2c_0\gamma_0 t$ by $2c_0 t$ in the integral, similarly if $\gamma_1 \geqslant 1$ we can replace $E(t, \gamma_1 w)$ by $E(t, w)$. If $\gamma_0 > 1$ and $0 < \gamma_1 < 1$ then adding to both sides $\int_{t\gamma_1}^{2c_0\gamma_0 t} (s^\alpha E(w, s))^r \frac{ds}{s}$ we have

$$\int_{\gamma_1 t}^\infty (s^\alpha E(s, w))^r \frac{ds}{s} \leqslant \left(cD + \frac{(2c_0\gamma_0)^{\alpha r} - \gamma_1^{\alpha r}}{\alpha r}\right) t^\alpha E(t, \gamma_1 w)^r,$$

or equivalently

$$\int_t^\infty (s^\alpha E(s, w))^r \frac{ds}{s} \leqslant \gamma_1^{-\alpha} \left(cD + \frac{(2c_0\gamma_0)^r - \gamma_1^r}{\alpha r}\right) t^\alpha E(t, w)^r.$$

This condition is equivalent, by Part 1, with $w \in G_{a, r}(\bar{X})$.

Finally to see 3, using the Holmstedt–Nilsson formula we get

$$E\left(t, \frac{\gamma_0}{2c_0} w; X_0, E_{\alpha, r}(\bar{X})\right) \leqslant \left(\int_{\gamma_0 t}^\infty (s^\alpha E(s, w))^r \frac{ds}{s}\right)^{1/r} \leqslant ct^\alpha E(t, \gamma_1 w)$$

and 2 applies. ∎

Remark 5. The referee has kindly shown to us an example proving that the previous result does not hold if $r < 0$. To see this select w such that $E(t, w) = t^{-\alpha}(\log |t|)^\beta$, for $t \geqslant 1$, α, $\beta > 0$. Then, $w \in G_{\alpha, r}$ for $r < -1/\beta$ but $w \notin G_{\alpha, r}$ for $r \geqslant -1/\beta$.

Remark 6. Since the E and K functionals can be obtained from each other by Legendre transformations it is not difficult to see the correspondence between K and E Gehring conditions. To fix ideas we analyze in detail the pair (L^1, L^∞). Suppose that f satisfies a K-Gehring condition (cf. [17]) of the form

$$\frac{K(t^{1/p}, f; L^p, L^\infty)}{t^{1/p}} \leqslant c\, \frac{K(t, f; L^1, L^\infty)}{t}. \tag{4.7}$$

Given $\delta > 0$ let $f = f_0 + f_1$, be a nearly optimal decomposition for the E-functional, that is if we let $s = E(t, f; L^\infty, L^1)$, we have $\|f_0\|_{L^\infty} \leqslant t$ and $s \leqslant \|f_1\|_{L^1} \leqslant (1+\delta)\, s$. Then

$$\begin{aligned} &K\left(\frac{t}{(1+\delta)\, s}, f; L^\infty, L^1\right) \\ &\qquad \leqslant \|f_0\|_{L_\infty} + \frac{t}{(1+\delta)\, s}\, \|f_1\|_{L^1} \leqslant 2t \end{aligned} \tag{4.8}$$

$$\begin{aligned} &K\left(\frac{(1+\delta)\, s}{t}, f; L^1, L^\infty\right) \\ &\qquad \leqslant 2s(1+\delta) \quad \left(\text{since } K(t, f; X_0, X_1) = tK\left(\frac{1}{t}, f; X_1, X_0\right)\right), \end{aligned} \tag{4.9}$$

combining with (4.7) we get

$$\frac{K\left(\left(\frac{(1+\delta)\, s}{t}\right)^{1/p}, f; L^p, L^\infty\right)}{\left(\frac{(1+\delta)\, s}{t}\right)^{1/p}} \leqslant c \left(\frac{(1+\delta)\, s}{t}\right)^{-1} 2s(1+\delta). \tag{4.10}$$

Therefore we can select a decomposition such that $f = g_0 + g_1$, and

$$\|g_0\|_{L^p} + \left(\frac{(1+\delta)\, s}{t}\right)^{1/p} \|g_1\|_{L^\infty} \leqslant c\, 2s(1+\delta) \left(\frac{(1+\delta)\, s}{t}\right)^{-1/p'},$$

thus

$$\|g_1\|_{L^\infty} \leqslant c\, 2s(1+\delta) \left(\frac{(1+\delta)\, s}{t}\right)^{-1} = c\, 2t,$$

and

$$E(2ct, f; L^\infty, L^p) \leqslant \|g_0\|_{L^p} \leqslant c\, 2s(1+\delta) \left(\frac{(1+\delta)\, s}{t}\right)^{-1/p'};$$

moreover since $s = E(t, f, L_\infty, L_1)$ we obtain

$$E(2ct, f; L^\infty, L^p) \leqslant 2c(1+\delta)^{1-1/p'} t^{1/p'} E(t, f, L^\infty, L^1)^{1-1/p'},$$

raising to the power p and letting $\delta \to 0$ we finally get

$$\begin{aligned} E(2ct, f; L^\infty, E_{p-1,1}(L^\infty, L^1)^p) &= E(2ct, f; L^\infty, L^p)^p \\ &\leqslant 2ct^{p-1} E(t, f; L^\infty, L^p), \qquad (4.11) \end{aligned}$$

which by Lemma 5(2) implies that $f \in G_{p-1,1}(L^\infty, L^1)$.

Following [17] the proof of Theorem 1 is based on the Holmstedt–Nilsson formula above and the following elementary Lemma on differential inequalities (cf. [18] for similar results.)

LEMMA 6. *Let $h(s)$ be a decreasing function, $\alpha > 0$, and suppose that $h_\alpha(s) = s^\alpha h(s)$, satisfies*

$$\int_t^\infty h_\alpha(s)^r \frac{ds}{s} \leqslant C h_\alpha(t)^r, \qquad t > t_0. \qquad (4.12)$$

Then there exists $\alpha' > \alpha$ such that for all $q > 0$, $t > t_0$,

$$\int_t^\infty h_{\alpha'}(s)^q \frac{ds}{s} \leqslant \tilde{C} h_{\alpha'}(t)^q.$$

Proof. We first show that (4.12) implies the existence of c, $\gamma > 0$ such that for $0 < x \leqslant y/2$ we have

$$y^c h_\alpha(y) \leqslant \gamma x^c h_\alpha(x). \qquad (4.13)$$

To prove this claim note that by (4.12) there exists $c \in (0, 1)$, such that

$$\frac{c}{t} \leqslant -\frac{\partial}{\partial t} \log\left(\int_t^\infty h_\alpha(s)^r \frac{ds}{s}\right) = \frac{h_\alpha(t)^r\, t^{-1}}{\int_t^\infty h_\alpha(s)^r \frac{ds}{s}}. \qquad (4.14)$$

Integrating (4.14) from x to $y/2$ we get

$$c \log \frac{y}{2x} \leqslant \log\left(\frac{\int_x^\infty h_\alpha(s)^r \frac{ds}{s}}{\int_{y/2}^\infty h_\alpha(s)^r \frac{ds}{s}}\right),$$

from where it readily follows that

$$\left(\frac{y}{2}\right)^c \int_{y/2}^{\infty} h_\alpha(s)^r \frac{ds}{s} \leqslant x^c \int_x^{\infty} h_\alpha(s)^r \frac{ds}{s} \leqslant C x^c h_\alpha(x)^r. \tag{4.15}$$

On the other hand, since h is decreasing,

$$\begin{aligned}\left(\frac{y}{2}\right)^c \int_{y/2}^{\infty} h_\alpha(s)^r \frac{ds}{s} &\geqslant \left(\frac{y}{2}\right)^c \int_{y/2}^{y} h_\alpha(s)^r \frac{ds}{s} = \left(\frac{y}{2}\right)^c \int_{y/2}^{y} s^{\alpha r} h(s)^r \frac{ds}{s} \\ &\geqslant \left(\frac{y}{2}\right)^c h(y)^r\, y^{\alpha r} \left(\frac{1-(1/2)^{\alpha r}}{\alpha r}\right). \end{aligned} \tag{4.16}$$

Combining (4.15) and (4.16) we finally obtain

$$y^c h_\alpha(y)^r \leqslant \gamma x^c h_\alpha(x)^r.$$

Let $q = r(1+\varepsilon)$, $\alpha' = \alpha + \theta$, where $\varepsilon > -1$ and $\theta \in (0, c/r)$, then

$$\begin{aligned} h_{\alpha'}(s)^q &= h(s)^{r(1+\varepsilon)}\, s^{(\alpha+\theta)\, r(1+\varepsilon)} = h(s)^{r(1+\varepsilon)}\, s^{\alpha r(1+\varepsilon)} s^{\theta r(1+\varepsilon)} \\ &= h_\alpha(s)^{r(1+\varepsilon)}\, s^{\theta r(1+\varepsilon)} s^{c(1+\varepsilon)} s^{-c(1+\varepsilon)} = (s^c h_\alpha(s)^r)^{1+\varepsilon}\, s^{(r\theta - c)(1+\varepsilon)}; \end{aligned}$$

thus

$$\begin{aligned} \int_x^{\infty} h_{\alpha'}(y)^q \frac{ds}{y} &= \int_x^{\infty} (y^c h_\alpha(y)^r)^{r+\varepsilon}\, y^{(r\theta - c)(1+\varepsilon)} \frac{dy}{y} \\ &= \int_x^{2x} (y^c h_\alpha(y)^r)^{1+\varepsilon}\, y^{(r\theta - c)(1+\varepsilon)} \frac{dy}{y} \\ &\quad + \int_{2x}^{\infty} (y^c h_\alpha(y)^r)^{1+\varepsilon}\, y^{(r\theta - c)(1+\varepsilon)} \frac{dy}{y} \\ &= I + II. \end{aligned}$$

To estimate II we apply (4.13) to obtain

$$II \leqslant (\gamma x^c h_\alpha(x)^r)^{1+\varepsilon} \int_{2x}^{\infty} y^{(r\theta - c)(1+\varepsilon) - 1}\, dy.$$

Since $r\theta - c < 0$, we get

$$II \leqslant (K x^c h_\alpha(x)^r)^{1+\varepsilon} \frac{(2x)^{(r\theta - c)(1+\varepsilon)}}{(c - r\theta)(1+\varepsilon)} = C' h_{\alpha'}(x)^{r(1+\varepsilon)}.$$

On the other hand,

$$I=\int_x^{2x} h(s)^{r(1+\varepsilon)}\, s^{(\alpha+\theta)\, r(1+\varepsilon)} \frac{dy}{y} \leqslant h(x)^{r(1+\varepsilon)} \int_x^{2x} s^{(\alpha+\theta)\, r(1+\varepsilon)} \frac{dy}{y}$$
$$\leqslant C'' h_{\alpha'}(x)^{r(1+\varepsilon)}.$$

The result follows. ∎

We are now ready to give the

Proof of Theorem 1. Our starting point is (4.6). By Lemma 5 this is equivalent to

$$\int_t^{\infty} (s^{\alpha} E(s, f))^r \frac{ds}{s} \leqslant C(t^{\alpha} E(t, f))^r.$$

By Lemma 6, with $h_{\alpha}(s)=s^{\alpha}E(f, s)$ we can select $\alpha'>\alpha$ such that for all $q>0$,

$$\left(\int_t^{\infty} (s^{\alpha'} E(s, f))^q \frac{ds}{s}\right)^{1/q} \leqslant C t^{\alpha'} E(t, f),$$

which again by Lemma 5 is equivalent to

$$E(t, f; X_0, E_{\alpha', q}(\bar{X})) \leqslant \tilde{c} t^{\alpha'} E(t, f),$$

as desired.

It is readily seen that elements of an approximation space that satisfy a Gehring condition belong, as should be expected, to a better approximation space. Indeed we have

THEOREM 5. *Let $\bar{X}=(X_0, X_1)$ be a pair of c_j-quasi-normed Abelian groups and let $f\in E_{\alpha, r}(X_0, X_1)$ be such that $f\in G_{a, r}$ then there exists $\alpha'>\alpha$ such that for all $q\geqslant r$*

$$\|f\|_{E_{\alpha', q}(X_0, X_1)} \leqslant c t_0^{\alpha'-\alpha} \|f\|_{E_{\alpha, r}(X_0, X_1)}.$$

Proof. Let $f\in G_{a, r}$ then, by Theorem 1, there exists $\alpha'>\alpha$ such that for all $t\geqslant t_0$

$$E(t, f; X_0, E_{\alpha', r}(\bar{X})) \leqslant \tilde{c} t^{\alpha'} E(t, f),$$

therefore by Lemma 5,

$$\left(\int_t^{\infty} (s^{\alpha'} E(s, f))^r \frac{ds}{s}\right)^{1/r} \leqslant c' t^{\alpha'} E(t, f).$$

Thus, for $t=t_0$ we have

$$\left(\int_{t_0}^{\infty} (s^{\alpha'}E(s,f))^r \frac{ds}{s}\right)^{1/r} \leqslant ct_0^{\alpha'} E(t,f_0)$$

$$= ct_0^{\alpha'-\alpha} E(t,f_0)\left(\alpha r \int_0^{t_0} s^{\alpha r}\frac{ds}{s}\right)^{1/r}$$

$$\leqslant ct_0^{\alpha'-\alpha}\left(\alpha r \int_0^{t_0} (s^{\alpha}E(s,f))^r \frac{ds}{s}\right)^{1/r}$$

$$\text{(since } E \text{ decreases)}$$

$$\leqslant ct_0^{\alpha'-\alpha}(\alpha r)^{1/r}\,\|f\|_{E_{\alpha,r}(X_0,X_1)}.$$

On the other hand,

$$\left(\int_0^{t_0} (s^{\alpha'}E(s,f))^r \frac{ds}{s}\right)^{1/r} = \left(\int_0^{t_0} (s^{\alpha'-\alpha}s^{\alpha}E(s,f))^r \frac{ds}{s}\right)^{1/r}$$

$$\leqslant t_0^{\alpha'-\alpha}\left(\int_0^{t_0} (s^{\alpha}E(s,f))^r \frac{ds}{s}\right)^{1/r}.$$

Combining these estimates we have

$$\|f\|_{E_{\alpha',r}(X_0,X_1)} \leqslant (c^r\alpha r+1)^{1/r}\, t_0^{\alpha'-\alpha}\,\|f\|_{E_{\alpha,r}(X_0,X_1)}.$$

This proves our result for $q=r$, if $q>r$ the result follows from the trivial inclusion

$$E_{\alpha',r}(X_0,X_1) \subset E_{\alpha',q}(X_0,X_1). \quad \blacksquare$$

We now prove Theorem 1 which provides us with an intrinsic characterization of $\bigcup G_{\alpha,r}$.

Proof of Theorem 2. Suppose that $f \in G_{\alpha,r}$, and let $\varepsilon>0$. By the proof of Theorem 1 there exists $\alpha'>\alpha$ such that for $t\geqslant t_0$ the function $t^{\alpha'}E(f,t)$ is almost decreasing. Therefore there exists $C>0$ such that for all $\xi>1$ we have

$$(t\xi)^{\alpha'}\,E(t,f\xi) \leqslant Ct^{\alpha'}E(t,f)$$

$$\frac{E(t,f\xi)}{E(t,f)} \leqslant \frac{C}{\xi^{\alpha}}\,\frac{1}{\xi^{\alpha'-\alpha}}.$$

Thus if we select $\xi > (\frac{C}{\varepsilon})^{1/(\alpha' - \alpha)}$, we have

$$\frac{E(t, f\xi)}{E(t, f)} \leqslant \frac{\varepsilon}{\xi^{\alpha}},$$

as we wished to show.

To prove the converse let $\varepsilon = 2^{-1}$, and select ξ so that (1.5) holds. Note that since $\xi > 1$ and E-functionals are decreasing, we can iterate (1.5) and obtain

$$\frac{E(t, \xi^n f)}{E(t, f)} \leqslant \frac{2^{-n}}{\xi^{n\alpha}}, \qquad n = 1, ... \tag{4.17}$$

Now,

$$\begin{aligned}\int_t^\infty s^{\alpha r - 1} E(s, f)^r \, ds &= \sum_{n=0}^{\infty} \int_{\xi^n t}^{\xi^{n+1} t} s^{\alpha r - 1} E(s, f)^r \, ds \\ &\leqslant \sum_{n=0}^{\infty} E(t, \xi^n f)^r \, t^{\alpha r} \xi^{n\alpha r} \left(\frac{\xi^{\alpha r} - 1}{\alpha r} \right) \\ &\leqslant \sum_{n=0}^{\infty} E(t, f)^r \frac{2^{-nr}}{\xi^{n\alpha r}} t^{\alpha r} \xi^{n\alpha r} \left(\frac{\xi^{\alpha r} - 1}{\alpha r} \right) \qquad \text{(by (4.17))}.\end{aligned}$$

Therefore,

$$\int_t^\infty s^{\alpha r - 1} E(s, f)^r \, ds \leqslant C t^{\alpha r} E(t, f)^r,$$

as we wished to show.

The condition in Theorem 2 does not depend on r, therefore we have obtained the following

Corollary 1. *There following are equivalent*

1. $f \in G_{\alpha, r}$ *for some* $r > 0$
2. $f \in G_{\alpha, r}$ *for all* $r > 0$.

5. EXAMPLES AND APPLICATIONS

5.1. *Reiteration of Gehring conditions.* As pointed out in the previous section, Theorem 1 applied to the pair (L^∞, L^p), gives a new proof of

Lemma 1. Let us apply Theorem 1 to the pair (L^∞, L^0). Since $E(t, w; L^\infty, L^0) = \lambda_w(t)$ (cf. Example 1(4.2)), we have $E_{p,1}(L^\infty, L^0) = (L^p)^p$. An application of Theorem 4 gives

$$C \int_{c_1 t}^{\infty} s^p \lambda_w(s) \frac{ds}{s} \leqslant E(t, w; L^\infty, (L^p)^p) \leqslant c \int_{c_2 t}^{\infty} s^p \lambda_w(s) \frac{ds}{s}.$$

It follows that $w \in G_{p,1}(L^\infty, L^0)$ iff there exist $C > 0$, $t_0 > 0$ such that for all $t \geqslant t_0$ we have

$$\int_t^\infty s^p \lambda_w(s) \frac{ds}{s} \leqslant C t^p \lambda_w(t).$$

The equivalence of the last condition and (1.4) was established in Theorem 3 (see in particular Condition 5). In other words, the content of Theorem 3 is that to prove Lemma 1 we can apply Theorem 1 using Gehring conditions with either (L^∞, L^0) or (L^∞, L^1) as our "initial pair". The reiteration formulae

$$L^1 = E_{1,1}(L^\infty, L^0), \qquad E_{p,1}(L^\infty, L^0) = (L^p)^p = E_{p-1,1}(L^\infty, L^1), \tag{5.1}$$

suggests that a general principle is behind this. Indeed, combining the method of proof of Theorem 3 with the Holmstedt–Nilsson formula we will show a general reiteration theorem for Gehring conditions. Let us first recall the following (known[6]) reiteration formulae for approximation spaces whose proof, for the sake of completeness, we shall present below.

Lemma 7. *Let $\bar{X} = (X_0, X_1)$ be a pair of c_j-quasi-normed Abelian groups then*

$$E_{\alpha-\beta, r}(X_0, E_{\beta, r}(\bar{X})) = E_{\alpha, r}(\bar{X}), \qquad r > 0, \quad \alpha > \beta > 0.$$

We can now state and prove a reiteration theorem for Gehring conditions.

Theorem 6. *Let $\bar{X} = (X_0, X_1)$ a pair of c_j-quasi-normed Abelian groups, $r > 0$, $\alpha > \beta > 0$. Then,*

$$G_{\alpha, r}(X_0, X_1) = G_{\alpha-\beta, r}(X_0, E_{\beta, r}(\bar{X})).$$

Proof. Suppose that $f \in G_{\alpha, r}(X_0, X_1)$, then there exist $c > 0$, $t_0 > 0$ such that

$$E(t, f; X_0, E_{\alpha, r}(\bar{X})) \leqslant c t^\alpha E(t, f), \qquad t \geqslant t_0.$$

[6] A proof can be obtained combining (3.11.5) and (7.1.7) in [3].

By Lemma 7 we can rewrite this inequality as

$$E(t, f; X_0, E_{\alpha-\beta, r}(X_0, E_{\beta, r}(\bar{X}))) \leqslant ct^{\alpha} E(t, f), \qquad t \geqslant t_0. \tag{5.2}$$

We estimate the right hand side of (5.2) using the fact that $E(t, f)$ is decreasing,

$$t^{\alpha} E(t, f) = t^{\alpha-\beta}(t^{\beta} E(t, f; X_0, X_1)) \leqslant ct^{\alpha-\beta} \left(\int_{t/2}^{t} (s^{\beta} E(t, f))^r \frac{ds}{s} \right)^{1/r}.$$

Estimating the left hand side of (5.2) from below using Holmstedt–Nilsson, and combining with the last inequality we find

$$\left(\int_{2c_0 t}^{\infty} (s^{\alpha-\beta} E(t, f; X_0, E_{\beta, r}(\bar{X})))^r \frac{ds}{s} \right)^{1/r}$$
$$\leqslant c' t^{\alpha-\beta} E\left(t, \frac{f}{4c_0}; X_0, E_{\beta, r}(\bar{X}) \right).$$

Therefore by Lemma 5-3 $f \in G_{\alpha-\beta, r}(X_0, E_{\beta, r}(\bar{X}))$.

Conversely, suppose that $f \in G_{\alpha-\beta, r}(X_0, E_{\beta, r}(\bar{X}))$, then for all $t \geqslant t_0$, we have

$$E(t, f; X_0, E_{\alpha, r}(\bar{X})) \leqslant \tilde{c} t^{\alpha-\beta} E(t, f; X_0, E_{\beta, r}(\bar{X})).$$

By the Holmstedt–Nilsson formula, and the fact that $(a+b)^r \leqslant a^r + b^r$ if $0 < r < 1$, or $2^{1-r}(a+b)^r \leqslant a^r + b^r$ if $r \geqslant 1$, we arrive to

$$\int_{2c_0 t}^{\infty} (s^{\alpha} E(s, f))^r \frac{ds}{s} \leqslant b'' \left\{ t^{\alpha r} E(t, f)^r + t^{(\alpha-\beta) r} \int_{t}^{\infty} (s^{\beta} E(s, f)) \frac{ds}{s} \right\}.$$

Adding $\int_t^{2c_0 t} (s^{\alpha} E(f, t))^t \frac{ds}{s}$ to both sides of the inequality and collecting terms we get

$$\int_{t}^{\infty} (s^{\alpha} E(s, f))^r \frac{ds}{s} \leqslant dt^{\alpha r} E(t, f)^r + b'' t^{(\alpha-\beta) r} \int_{t}^{\infty} (s^{\beta} E(s, f))^r \frac{ds}{s}$$

$$\int_{t}^{\infty} s^{\alpha r} \left(1 - b'' \frac{t^{(\alpha-\beta), r}}{s^{(\alpha-\beta) r}} \right) E(s, f)^r \frac{ds}{s} \leqslant dt^{\alpha r} E(t, f)^r.$$

At this point note that $(1 - b'' \frac{t^{(\alpha-\beta) r}}{s^{(\alpha-\beta) r}}) \geqslant \frac{1}{2}$ if $s \geqslant (2b'')^{1/(\alpha-\beta) r}\, t$, therefore

$$\int_{(2b'')^{1/(\alpha-\beta) r}\, t}^{\infty} s^{\alpha r} E(s, f)^r \frac{ds}{s} \leqslant 2\, dt^{\alpha r} E(t, f)^r.$$

Therefore once again by Lemma 5(3) we see that $f \in G_{\alpha, r}(X_0, X_1)$. ∎

We now give the proof of Lemma 7.

Proof. Let $f \in E_{\alpha-\beta, r}(X_0, E_{\beta, r}(\bar{X}))$. By Theorem 4 we have

$$\begin{aligned}\|f\|^r_{E_{\alpha-\beta, r}(X_0, E_{\beta, r}(\bar{X}))} &= \int_0^\infty (s^{\alpha-\beta} E(t, f; X_0, E_{\beta, r}(\bar{X})))^r \frac{ds}{s} \\ &\geqslant c \int_0^\infty \left(s^{\alpha-\beta}\left(\int_{2c_0 s}^\infty (z^\beta E(z, f))^r \frac{dz}{z}\right)^{1/r}\right)^r \frac{ds}{s} \\ &\geqslant c \int_0^\infty \left(s^{(\alpha-\beta) r} \int_{2c_0 s}^{4c_0 s} (z^\beta E(z, f))^r \frac{dz}{z}\right) \frac{ds}{s} \\ &\geqslant c \int_0^\infty s^{(\alpha-\beta) r} (2c_0 s)^{\beta r} E(c, 4f_0 f)^r \frac{ds}{s} \\ &\geqslant c' \|f\|^r_{E_{\alpha, r}(\bar{X})}.\end{aligned}$$

Conversely suppose that $f \in E_{\alpha, r}(\bar{X})$, then by Theorem 4 we find

$$\begin{aligned}&\|f\|_{E_{\alpha-\beta, r}(X_0, E_{\beta, r}(\bar{X}))} \\ &\quad = \left(\int_0^\infty (s^{\alpha-\beta} E(s, f; X_0, E_{\beta, r}(\bar{X})))^r \frac{ds}{s}\right)^{1/r} \\ &\quad \leqslant c \left(\int_0^\infty s^{(\alpha-\beta) r} \left(s^\beta E(s, f) + \left(\int_s^\infty (z^\beta E(z, f))^r \frac{dz}{z}\right)^{1/r}\right)^r \frac{ds}{s}\right)^{1/r} \\ &\quad \leqslant c \left(\left(\int_0^\infty (s^\alpha E(s, f))^r \frac{ds}{s}\right)^{1/r}\right. \\ &\qquad \left. + \left(\int_0^\infty \left(s^{(\alpha-\beta) r} \int_s^\infty (z^\beta E(z, f))^r \frac{dz}{z}\right) \frac{ds}{s}\right)^{1/r}\right).\end{aligned}$$

Integrating by parts the right-most integral we get

$$\int_0^\infty \left(s^{(\alpha-\beta) r} \int_s^\infty (z^\beta E(z, f))^r \frac{ds}{z}\right) \frac{ds}{s} = \int_0^\infty s^{\alpha r} E(s, f)^r \frac{ds}{s},$$

since the integrated term vanishes on account of the fact that $f \in E_{\alpha, r}(\bar{X})$. Thus, we find that

$$\begin{aligned}\|f\|_{E_{\alpha-\beta, r}(X_0, E_{\beta, r}(\bar{X}))} &\leqslant c \left(\left(\int_0^\infty (s^\alpha E(s, f))^r \frac{ds}{s}\right)^{1/r} + \left(\int_0^\infty s^{\alpha r} E(s, f)^r \frac{ds}{s}\right)^{1/r}\right) \\ &\leqslant c \|f\|_{E_{\alpha, r}(\bar{X})},\end{aligned}$$

as we wished to show.

5.2. *Reverse Chebyshev inequalities.* In this section we consider "Generalized Reverse Chebyshev Inequalities" (cf. [5]) of the form

$$\int_{\{w>t\}} w(s)\,ds \leqslant ct^{\theta} \int_{\{w>t\}} w^{1-\theta}(s)\,ds, \qquad \theta \in (0, 1]. \tag{5.3}$$

Recall that for $0<\theta\leqslant 1$ the usual Chebyshev, and easily verified, inequalities state

$$t\lambda_w(t) \leqslant t^{\theta} \int_{\{w>t\}} w^{1-\theta}(s)\,ds \leqslant \int_{\{w>t\}} w(s)\,ds. \tag{5.4}$$

And easy application of Hölder's inequality shows that (5.3) is equivalent to

$$\int_{\{w>t\}} w(s)\,ds \leqslant ct\lambda_w(t). \tag{5.5}$$

To see that (5.5) has a self improving property, first note that by Lemma 3(2.1)

$$t\lambda_w(t) + \int_t^{\infty} \lambda_w(s)\,ds = \int_{\{w>t\}} w(s)\,ds \leqslant ct\lambda_w(t)$$

$$\int_t^{\infty} \lambda_w(s)\,ds \leqslant (c-1)\,t\lambda_w(t),$$

which in terms of E-functional inequalities means that

$$E(t, w; L^{\infty}, L^1) = E(t, w; L^{\infty}, E_{1,1}(L^{\infty}, L^0)) \leqslant (c-1)\,tE(t, w; L^{\infty}, L^0).$$

It follows that $w \in G_{1,1}(L^{\infty}, L^0)$, thus by Theorem 5 there exists $\varepsilon>0$ such that $w \in G_{1+\varepsilon,1}(L^{\infty}, L^0)$. Applying Theorem 5 we have the result of [5]: if $w \in L^1 = E_{1,1}(L^{\infty}, L^0)$ then $w \in E_{1+\varepsilon,1}(L^{\infty}, L^0) = (L^{1+\varepsilon})^{1+\varepsilon}$. We further have

$$(\|w\|_{L^{1+\varepsilon}})^{1+\varepsilon} \leqslant ct_0^{\varepsilon}\,\|f\|_{L^1}.$$

Therefore if (5.5) holds for $t \geqslant t_0 = \|f\|_{L^1}$, we get

$$\|w\|_{L^{1+\varepsilon}} \leqslant c\,\|f\|_{L^1}.$$

Compare with [5]. The result we presented here is stronger in as much as it shows the improvement at the level of the E-functionals as well:

$$\int_t^{\infty} s^{\varepsilon}\lambda_w(s)\,ds \leqslant \tilde{c}t^{1+\varepsilon}\lambda_w(t).$$

In view of Lemma 5 we also can also treat inequalities of the form

$$\int_{\{w>t\}} w(s)\,ds \leqslant ct^{\theta} \int_{\{w>\gamma t\}} w^{1-\theta}(s)\,ds, \qquad 0<\gamma<1.$$

Remark 7. We can also consider inequalities of the form

$$t^{\theta} \int_{\{w>t\}} w^{1-\theta}(s)\,ds \leqslant ct\lambda_w(t). \tag{5.6}$$

By Lemma 3(2.1) and the Holmstedt–Nilsson formula (5.6) is equivalent to

$$E(w,t;L^{\infty},L^{1-\theta}) = E(w,t;L^{\infty},E_{1-\theta,1}(L^{\infty},L^{0})) \leqslant \tilde{c}t^{1-\theta}E(w,t;L^{\infty},L^{0}).$$

Thus these inequalities also have the self improving property.

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